HW to be graded: 3.3.11, 3.3.13, 3.4.1, 3.4.9a, 3.5.8.

Exam Problem 3: (a_n) a sequence satisfying $\forall x \in (0,1) \cap \mathbb{Q}$,
$\exists n \in \mathbb{N}$, $a_n = x$.

(a) Why does such a sequence exist?

b/c $(0,1) \cap \mathbb{Q}$ is countable.

(b) Let A be the set of y such that for some subsequence (a_{n_k}) of (a_n), $\lim_{k\to\infty} a_{n_k} = y$. Find A and prove it to be the correct set.

$A = [0,1]$.

Why: Easy to see $A =$ the set of limit points of $\mathbb{Q} \cap (0,1)$.

Since there is no isolated point in $\mathbb{Q} \cap (0,1)$.

(b/c $\forall a<b$, $\exists r$, $r \in (a,b)$)

$$A = \overline{\mathbb{Q} \cap (0,1)} = [0,1]$$

(b/c $\mathbb{Q} \cap (0,1)$ is dense in $[0,1]$)

Exam Problem 6a. For the (a_n) above, find $\limsup_{n\to\infty} a_n$

<u>Recall</u>: $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} y_n$, where $y_n = \sup\{a_m : m \geq n\}$.

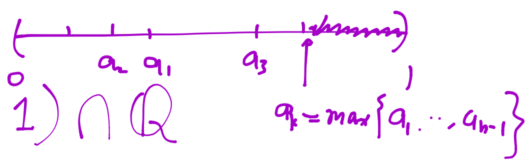
For this (a_n), $y_n = 1$ for every $n \in \mathbb{N}$.

Why? $\{a_m : m \geq n\} = [\mathbb{Q} \cap (0,1)] \setminus \{a_1, a_2, \cdots, a_{n-1}\}$

$\max\{a_1, a_2, \cdots, a_{n-1}\} < 1$.

$\Rightarrow \{a_m : m \geq n\} \supseteq (\max\{a_1, \cdots, a_{n-1}\}, 1) \cap \mathbb{Q}$

$\Rightarrow \sup\{a_m : m \geq n\}$



$\geq \sup(\max\{a_1, \cdots, a_{n-1}\}, 1) \cap \mathbb{Q}$

$= 1$.

$\sup\{a_m : m \geq n\}$ is obviously ≤ 1

$\Rightarrow y_n = \sup\{a_m : m \geq n\} = 1$

$\Rightarrow \limsup_{n\to\infty} a_n = \lim_{n\to\infty} y_n = 1$.

$f: A \to \mathbb{R}$, c is a limit point of A.

$\lim_{x\to c} f(x) = L$: $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in A, 0 < |x-c| < \delta, |f(x)-L| < \varepsilon.$

Rmk: 1. c does not have to be in A.

2. We only care about the behavior of $f(x)$ NEAR $x=c$.

The value of f at $x=c$ is irrelevant.

Topologically: $\lim_{x\to c} f(x) = L$ means $\forall$ ε-nbhd of L, say $V_\varepsilon(L)$

$\exists$ δ-nbhd of c, say $V_\delta(c)$

$\forall x \in V_\delta(c) \cap A$, $x \neq c$. $f(x) \in V_\varepsilon(L)$

Example: $f(x) = \dfrac{2x}{3x-1}$, Verify by definition that $\lim_{x\to 1} f(x) = 1$.

$$f(x) - 1 = \frac{2x}{3x-1} - 1 = \frac{2x-3x+1}{3x-1} = \frac{-x+1}{3x-1}$$

$$|f(x)-1| = \frac{|x-1|}{|3x-1|}$$

Since $x \to 1$. $3x-1 \to 2$. i.e., I can find some δ, s.t.

$x \in (1-\delta, 1+\delta) \setminus \{1\}$. $3x-1 \geq 1$

δ can be chosen $\frac{1}{3}$.

For $\delta_1 = \frac{1}{3}$, $\forall x \in (1-\delta_1, 1+\delta_1) \setminus \{1\}$.

$$3x-1 > 3\cdot(1-\delta_1) - 1 = 3\cdot(1-\tfrac{1}{3}) - 1 = 1$$

$$\Rightarrow |f(x)-1| = \frac{|x-1|}{3x-1} < \frac{|x-1|}{1}$$

If $\delta_2 < \varepsilon$, then $x \in (1-\delta_2, 1+\delta_2) \setminus \{1\}$. $|x-1| < \varepsilon$.

So for $\delta = \min(\delta_1, \delta_2)$, both inequalities hold. i.e. $x \in (1-\delta, 1+\delta) \setminus \{1\}$

$$|f(x) - 1| = \frac{|x-1|}{|3x-1|} = \frac{|x-1|}{3x-1} < \frac{|x-1|}{1} < \varepsilon.$$

Rmk: Inequalities are central to analysis.

Functional Limits in sequential form.

$f: A \to \mathbb{R}$, c limit point of A.

$\lim_{x \to c} f(x) = L \iff \forall (x_n) \subseteq A,\ x_n \neq c,\ (x_n) \to c,\ (f(x_n)) \to L$

i.e. If (x_n) approaches to c, no matter how, $f(x_n)$ should approach to L

Algebraic Limit Theorem: $f, g: A \to \mathbb{R}$, c is a limit point

$$\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M$$

(i)

(ii)

(iii)

(iv) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided $M \neq 0$

Divergence Criterion for Functional limits.

$f: A \to \mathbb{R}$, c limit point of A.

If $(x_n), (y_n) \subseteq A$; $x_n \neq c$, $y_n \neq c$, $\forall n$; s.t.

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = c, \text{ but } \lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n)$$

Then $\lim_{x \to c} f(x)$ DNE.

Example: $f(x) = \sin\frac{1}{x}$.

Example: $f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$ Dirichlet Function.

Does $\lim_{x\to 1} f(x)$ exist?

Let (r_n) be a sequence of rat'l #s s.t. $(r_n) \to 1$.

Since $f(r_n) = 1$, $\forall n$, $\Rightarrow$ $f(r_n) \to 1$, $n \to \infty$.

Let (s_n) be a sequence of irrat'l #s s.t. $(s_n) \to 1$.

Since $f(s_n) = 0$, $\forall n$, $\Rightarrow$ $f(s_n) \to 0$, $n \to \infty$.

By the divergence Criterion, $\lim_{x\to 1} f(x)$ DNE.

A function $f: A \to \mathbb{R}$ is continuous at a point $c \in A$,

if $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (c-\delta, c+\delta), |f(x) - f(c)| < \varepsilon$.

If f is continuous at every point in the domain A, then f is continuous on A.

Topological Version: $f: A \to \mathbb{R}$ is continuous if $\forall O \subseteq \mathbb{R}$ open, $f^{-1}(O)$ is open.

$$f^{-1}(O) = \{x \in A : f(x) \in O\}.$$

Theorem: TFAE.

(i) f is continuous at $x = c$

(ii) $\forall V_\varepsilon(f(c))$, $\exists V_\delta(c)$, s.t. $x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$.

(iii) $(x_n) \to c \Rightarrow (f(x_n)) \to f(c)$

Criterion for discontinuity: $f: A \to \mathbb{R}$, c is a limit point of A.

If for some $(x_n) \to c$, $f(x_n) \not\to f(c)$.

Then f is not continuous at c.

<u>Example</u>: The Dirichlet function is not continuous at any $c \in \mathbb{R}$.

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More detailed notes will be posted next week

My solutions to 3.3.11.

(a) $\mathbb{N}$ is not compact.

Consider the open set $O_n = (n - \frac{1}{2}, n + \frac{1}{2})$.

$\{O_n\}$ is an open cover.

Any finitely many O_n's could only cover finitely many elements in $\mathbb{N}$.

(b) $\mathbb{Q} \cap [0, 1]$ is not compact.

Consider the open set $O_n = (-\frac{1}{2}, \frac{1}{\sqrt{2}} - \frac{1}{n}) \cup (\frac{1}{\sqrt{2}} + \frac{1}{n}, \frac{3}{2})$.

$\bigcup_{n=1}^{\infty} O_n = (-\frac{1}{2}, \frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \frac{3}{2}) \supseteq \mathbb{Q} \cap [0, 1]$

So $\{O_n\}_{n=2}^{\infty}$ is an open cover.

Any finitely many O_n's won't cover $\mathbb{Q} \cap [0, 1]$.

$[0, 1] \setminus \bigcup_{i=1}^{k} O_{n_i} = [\frac{1}{\sqrt{2}} - \frac{1}{n_k}, \frac{1}{\sqrt{2}} + \frac{1}{n_k}]$

By density theorem, $\exists\, q \in \mathbb{Q} \cap [0,1]$ lying outside of $\bigcup_{i=1}^{k} O_{n_i}$.

(c) The Cantor Set is compact.

(d) $\left\{1+\frac{1}{2^2}+\cdots+\frac{1}{n^2} : n \in \mathbb{N}\right\}$ is not compact.

Consider $O_n = \left(\frac{1}{2},\, 1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)$

$\forall n,\ 1+\frac{1}{2^2}+\cdots+\frac{1}{n^2} \in O_{n+1}.\ \Rightarrow \forall n,\ 1+\frac{1}{2^2}+\cdots+\frac{1}{n^2} \in \bigcup_{n=1}^{\infty} O_n$

So $\{O_n\}_{n=1}^{\infty}$ is an open cover.

Any finitely many O_n's cannot cover $\left\{1+\frac{1}{2^2}+\cdots+\frac{1}{n^2} : n \in \mathbb{N}\right\}$

(e) This set is compact.